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Structure of $\mathbf{A}^2$-fibrations over one-dimensional noetherian domains

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Abstract

Let R be a one-dimensional noetherian domain containing the field $\mathbb{Q}$ of rational numbers. Let A be an $\mathbf{A}^2$-fibration over R. Then there exists $H \in A$ such that A is an $\mathbf{A}^1$-fibration over $R[H]$. As a consequence, if $\Omega_{A/R}$ is free then $A = R^{[2]}$.

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1. Introduction

Let R be a commutative ring with unity. Let $R^{[n]}$ denote a polynomial ring in n variables over R. For a prime ideal P of R, $K(P)$ denotes the field R_P/PR_P. An R-algebra A is said to be an $\mathbf{A}^r$-fibration over R if

(i) A is finitely generated over R.
(ii) A is flat over R.
(iii) $A \otimes_R K(P) = K(P)^{[r]}$ for every prime ideal P of R.

If R is a discrete valuation ring containing $\mathbb{Q}$ (the field of rationals) and A is an $\mathbf{A}^2$-fibration over R, then in [9, Theorem 1] Sathaye has proved that $A = R^{[2]}$. As a consequence, if A is an $\mathbf{A}^2$-fibration over a Dedekind ring R containing $\mathbb{Q}$, then by the result of Bass–Connell–Wright [4, 4.4] $A \cong Sym_R(E)$ for a finitely generated projective R-module E of rank 2. On the other hand, if a discrete valuation ring R does not contain $\mathbb{Q}$ then Asanuma [2, 5.1] has shown that there exists an $\mathbf{A}^2$-fibration A over R such that $A \ncong R^{[2]}$.

In view of the above mentioned results, it is natural to ask: *How does an* $\mathbf{A}^2$*-fibration arise over an arbitrary one-dimensional noetherian domain R containing* $\mathbb{Q}$? An obvious

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way of constructing $\mathbf{A}^2$-fibrations over R is to take the tensor product of $\mathbf{A}^1$-fibrations B and C over R. Let us call such $\mathbf{A}^2$-fibrations decomposable. Then one would like to know whether all $\mathbf{A}^2$-fibrations over a one-dimensional noetherian domain R (containing $\mathbb{Q}$) are decomposable. Example 3.12 of this paper shows that not all $\mathbf{A}^2$-fibrations are decomposable.

Another way to construct an $\mathbf{A}^2$-fibration over R is to take an $\mathbf{A}^1$-fibration B over R (e.g. $B = R^{[1]}$) and take an $\mathbf{A}^1$-fibration A over B. In this paper we show that this is the only way to construct $\mathbf{A}^2$-fibrations over R. More precisely, we prove (see Theorem 3.8):

Theorem. *Let R be a one-dimensional noetherian domain containing the field $\mathbb{Q}$ of rational numbers. Let A be an $\mathbf{A}^2$-fibration over R. Then there exists $H \in A$ such that A is an $\mathbf{A}^1$-fibration over $R[H]$.*

As a consequence of this theorem we show that (see Corollary 3.9) if $\Omega_{A/R}$ is extended from R then $A \cong Sym_R(E)$. In particular, if R is local, then $A = R^{[2]}$. Thus, $\Omega_{A/R}$ being not free is the only obstruction for the result of Sathaye to be not true for an arbitrary local domain R of dimension 1.

In Section 2 we set up notations and quote some results for later use. In Section 3 we prove our main theorems.

2. Preliminaries

Throughout this paper all rings will be commutative, noetherian with unity. We now set up notations and state some results for later use.

$\mathbb{Q}$: field of rational numbers.

For a commutative ring R,

$R^{[n]}$: polynomial ring in n variables over R.

For an R-module E,

$Sym_R(E)$: symmetric algebra of E over R.

For a prime ideal P of R,

$K(P)$: R_P/PR_P.

For a finitely generated R-algebra A,

$\Omega_{A/R}$: universal module of R-differentials of A.

For elements $F, G \in R[X, Y] = R^{[2]}$,

$\mathbf{J}_{(X,Y)}(F, G)$: $(\partial F/\partial X)(\partial G/\partial Y) - (\partial F/\partial Y)(\partial G/\partial X)$

Definition 2.1. An R-algebra A is said to be an $\mathbf{A}^r$*-fibration* over R if the following hold:

(i) A is finitely generated over R.
(ii) A is R-flat.
(iii) $A \otimes_R K(P) = K(P)^{[r]}$ for every prime ideal P of R.

Definition 2.2. An $\mathbf{A}^r$-fibration A over R is said to be *decomposable* if there exists an $\mathbf{A}^l$-fibration B over R and an $\mathbf{A}^m$-fibration C over R for positive integers l and m such that $A \cong B \otimes_R C$ as R-algebras.

Definition 2.3. Let A be a ring and R a subring of A. R is said to be a *retract* of A if there exists an R-algebra homomorphism $\alpha : A \to R$.

We will now quote some results which will be needed in this paper. We begin with the following result [2, 3.4]:

Theorem 2.4. *If A is an $\mathbf{A}^r$-fibration over a commutative noetherian ring R, then $\Omega_{A/R}$ is a finitely generated projective A-module of (constant) rank r and A is (upto an R-isomorphism) an R-subalgebra of $R^{[m]}$ for some m such that*

$$A^{[m]} \cong Sym_{R^{[m]}}(R^{[m]} \otimes_A \Omega_{A/R}):$$

as R-algebras. Therefore A is a retract of $R^{[n]}$ for some n.

Next we state a result which follows from [4, 4.4]:

Theorem 2.5. *Let R be a noetherian commutative ring and let A be a finitely generated R-algebra such that $A_M \cong R_M^{[r]}$ as R_M-algebras for all maximal ideals M of R. Then, $A \cong Sym_R(E)$ for some finitely generated projective R-module E of (constant) rank r.*

The following result is due to Hamann [7, 2.8]:

Theorem 2.6. *Let R be a noetherian ring containing $\mathbb{Q}$. Then, $R^{[1]}$ is R-invariant, that is, if A is an R-algebra such that $A^{[m]} = R^{[m+1]}$ as R-algebras, then $A = R^{[1]}$.*

As a consequence of Theorems 2.4–2.6 we derive the following:

Proposition 2.7. *Let R be a noetherian ring containing $\mathbb{Q}$. Let A be an $\mathbf{A}^1$-fibration over R such that $\Omega_{A/R}$ is extended from R. Then, $A \cong Sym_R(L)$ as R-algebras for some finitely generated projective R-module L of (constant) rank* 1:

The next theorem is due to Sathaye [9, Theorem 1]:

Theorem 2.8. *Let R be a discrete valuation ring containing $\mathbb{Q}$. Let A be an $\mathbf{A}^2$-fibration over R. Then, $A = R^{[2]}$.*

The following theorem is the famous epimorphism theorem of Abhyankar and Moh [1]:

Theorem 2.9. *Let K be a field of characteristic 0. Let $f(Z)$, $g(Z)$ be two elements of $K[Z]$ $(=K^{[1]})$ such that $K[Z] = K[f(Z), g(Z)]$ where $\deg_Z f(Z) = m > 0$, $\deg_Z g(Z) = n > 0$. Then, either m divides n or n divides m. As a consequence, if $W \in K[X, Y]$ $(=K^{[2]})$ be such that $K[X, Y]/(W) = K^{[1]}$ as K-algebras, then $K[X, Y] = K[W]^{[1]}$.*

We conclude this section by stating a result from [6, 3.1]:

Theorem 2.10. *Let R be a noetherian ring of finite (Krull) dimension d. Let E be a finitely generated projective $R^{[n]}$-module of rank $\geq d + 1$. Then, there exists a surjective $R^{[n]}$-linear homomorphism $\psi : E \to R^{[n]}$. In particular, if $d = 1$, and E is a finitely generated projective $R^{[n]}$-module of constant rank m with $\bigwedge^m E \cong R^{[n]}$ (as $R^{[n]}$-modules), then E is free (of rank m).*

3. Main theorems

In this section we shall prove our main theorems (Theorems 3.8 and 3.11). For the proof of these theorems we need some lemmas. We begin with

Lemma 3.1. *Let K be a field of characteristic 0. Let $K[U, V] = K^{[2]}$ and let $f(U) \in K[U]$ and $g(V) \in K[V]$ be such that $\deg_U f(U) > 0$ and $\deg_V g(V) > 0$. Let $h(U, V) = f(U) + g(V)$. Then $K[U, V]/(h(U, V))$ is a reduced ring.*

Proof. Since $\partial h(U, V)/\partial U = f'(U)$ $(= df(U)/dU)$, $\partial h(U, V)/\partial V = g'(V)$ $(= dg(V)/dV)$ and $f'(U)$, $g'(V)$ are non-zero, we have $\operatorname{ht}(\partial h(U, V)/\partial U, \partial h(U, V)/\partial V) \geq 2$. Therefore the result follows. □

Lemma 3.2. *Let T be a ring containing $\mathbb{Q}$ and let $\alpha_1, \ldots, \alpha_{n+1}$ be $n + 1$ distinct rational numbers. Let $F(X, Y) \in T[X, Y]$ be a homogeneous polynomial in X and Y of degree n. Then there exist $t_1, \ldots, t_{n+1} \in T$ such that $F = \sum_{i=1}^{n+1} t_i(X + \alpha_i Y)^n$.*

Proof. Since it is enough to prove the result for monomials of the type $X^l Y^m$ $(l + m = n)$, without loss of generality we can assume that $T = \mathbb{Q}$. In that case it is enough to show that the (homogeneous) polynomials $(X + \alpha_i Y)^n$, $1 \leq i \leq n + 1$, are linearly independent over $\mathbb{Q}$.

Let $\beta_1, \ldots, \beta_{n+1} \in \mathbb{Q}$ be such that

$$\sum_{i=1}^{n+1} \beta_i (X + \alpha_i Y)^n = 0.$$

The above equation implies that $\sum_{i=1}^{n+1} \alpha_i^j \beta_i = 0$ for all j, $0 \le j \le n$. Therefore we get the following matrix equation:

$$[\beta_1, \ldots, \beta_{n+1}]\Lambda = [0, \ldots, 0].$$

where Λ is the $(n+1 \times n+1)$ Vandermonde matrix whose (i,j)th entry is α_i^{j-1}. Since $\alpha_1, \ldots, \alpha_{n+1}$ are all distinct, the matrix Λ is invertible. Therefore $\beta_i = 0$ for $1 \le i \le n+1$.

Thus the polynomials $(X + \alpha_i Y)^n$, $1 \le i \le n+1$, are linearly independent over $\mathbb{Q}$. □

Lemma 3.3. *Let R be a one-dimensional noetherian domain containing $\mathbb{Q}$. Let A be an $\mathbb{A}^2$-fibration over R. Then there exists a finite birational extension T of R and a finitely generated projective T-module L of rank 1 such that $A \otimes_R T = Sym_T(L)^{[1]}$.*

Proof. Let K denote the quotient field of R and $\tilde{R}$ denote the normalization of R.

Since $A \otimes_R K = K^{[2]}$ and A is finitely generated over R, there exists $t(\neq 0) \in R$ such that $A[1/t] = R[1/t]^{[2]}$. If t is invertible then there is nothing to prove. So we assume that t is not invertible.

Let $S = 1 + tR$. Then R_S is semilocal and hence $\tilde{R}_S$ is a semilocal principal ideal domain. Therefore $(A \otimes_R \tilde{R})_S = \tilde{R}_S^{[2]}$ by Theorems 2.5 and 2.8. Hence as before there exists $s \in S$ such that $A \otimes_R \tilde{R}[1/s] = \tilde{R}[1/s]^{[2]}$. Since A is flat over R and every finite birational extension of $R[1/s]$ is of the type $R'[1/s]$ where R' is a finite birational extension of R, it follows that there exists a finite birational extension T of R such that $A \otimes_R T[1/s] = T[1/s]^{[2]}$. Moreover, as $A[1/t] = R[1/t]^{[2]}$, we have $A \otimes_R T[1/t] = T[1/t]^{[2]}$.

Thus $A \otimes_R T[1/s] = T[1/s]^{[2]}$, $A \otimes_R T[1/t] = T[1/t]^{[2]}$ and $tR + sR = R$. Therefore by Theorem 2.5, there exists a finitely generated projective T-module E of rank 2 such that $A \otimes_R T = Sym_T(E)$. Since rank $E = 2 > \dim T$, by a theorem of Serre [3, Corollary 2.7, p. 173], $E \cong L \oplus T$. Therefore $A \otimes_R T = Sym_T(L)^{[1]}$. □

Lemma 3.4. *Let T be a noetherian ring and let I and N be proper ideals of T such that $I + N = T$. Let L be a finitely generated projective T-module of (constant) rank 1 such that L/IL is free over T/I and let $X \in L/IL$ be a generator of L/IL. Let $B = Sym_T(L)$ and $D = B[Y] (= B^{[1]})$. Then $D/ID = (T/I)[X, Y] (= (T/I)^{[2]})$. Moreover if $X_1, Y_1 \in D/ID$ be such that*

$$(X_1, Y_1) = (X, Y + f(X)) \quad or \quad (X + g(Y), Y),$$

where $f(X) \in (T/I)[X]$ and $g(Y) \in (T/I)[Y]$, then the T/I-algebra automorphism θ of D/ID $(= (T/I)[X, Y])$ given by $\theta(X) = X_1$, $\theta(Y) = Y_1$ can be lifted to a T-algebra automorphism σ of D such that $\sigma \equiv$ identity (mod ND).

Proof. Since $I + N = T$, there exists an element $s \in N$ such that $1 - s \in I$. Now, as $D/ID = (B/I)[Y] = Sym_{T/I}(L/IL)[Y]$ and L/IL is a free T/I-module of rank 1 with a generator X, it is easy to see that $D/ID = (T/I)[X, Y]$ $(= T/I^{[2]})$.

If $X_1 = X$ and $Y_1 = Y + f(X)$ with $f(X) \in (T/I)[X]$ $(= B/IB)$ then let $b \in B$ be such that $\bar{b}$ (the image of b in B/IB) $= f(X)$. Let σ be a B-algebra (and hence a T-algebra) automorphism of D $(= B[Y])$ such that $\sigma(Y) = Y + sb$. It is clear that σ is a lift of θ and $\sigma \equiv$ identity (mod ND).

Now suppose that $X_1 = X + g(Y)$ and $Y_1 = Y$. Let $\psi : L/IL \to (T/I)[Y]$ be a T/I-module homomorphism given by $\psi(X) = g(Y)$. Then since L is T-projective, ψ can be lifted to a T-module homomorphism $\eta : L \to T[Y]$. Let $\phi = s\eta$ and let $\rho : L \to D$ be a T-module homomorphism given by $\rho(e) = e + \phi(e)$ for $e \in L$. Since $B = Sym_T(L)$ and D is commutative, ρ gives rise to a T-algebra homomorphism $\sigma : B \to D$ $(= B[Y])$. We extend σ to a T-algebra endomorphism of D (which we still denote by σ) by putting $\sigma(Y) = Y$. Since $D = Sym_T(L)[Y]$, and $\phi(e) \in T[Y]$ $\forall e \in L$, it is easy to see that σ is a locally (i.e. taking all local rings of T) automorphism. Hence σ is a T-algebra automorphism of D such that σ is a lift of θ. Since $s \in N$, it is obvious that $\sigma \equiv$ identity (mod ND).

We here note that $D = Sym_T(\sigma(L))[\sigma(Y)]$ and $\sigma(L)/I\sigma(L)$ is a free T/I-module with a generator X_1. □

The following lemma is an easy consequence of Theorem 2.9:

Lemma 3.5. *Let K be a field of characteristic 0 and let $W \in K[X, Y]$ $(= K^{[2]})$ be such that $K[X, Y] = K[W]^{[1]}$. Then there exists a sequence (X_i, Y_i), $1 \le i \le l$, of pairs of variables in $K[X, Y]$ such that $Y_l = W$ and*

$$(X_0, Y_0) = (X, Y),$$

$$(X_i, Y_i) = (X_{i-1}, Y_{i-1} + f_i(X_{i-1})) \quad \text{or} \quad (X_{i-1} + g_i(Y_{i-1}), Y_{i-1}),$$

where $f_i(X_{i-1}) \in K[X_{i-1}]$, $g_i(Y_{i-1}) \in K[Y_{i-1}]$ for $1 \le i \le l$.

Proof. Since $K[X, Y] = K[W]^{[1]}$, we have $K[X, Y]/(W) \cong K(Z)$. Let $\Psi : K[X, Y] \to K[Z]$ be a surjective K-algebra homomorphism such that $\ker(\Psi) = (W)$.

If $\Psi(Y) \in K$ then there exists $\delta \in K$ such that $\Psi(Y - \delta) = 0$ and hence there exists $\beta \in K^*$ such that $W = \beta(Y - \delta)$. Let $\alpha = \beta^{-1}$ and $\lambda = 1 - \alpha$. Now consider the pairs (X_i, Y_i), $1 \le i \le 4$, of variables defined as follows:

$$(X_1, Y_1) = (X, Y - \delta), \qquad (X_2, Y_2) = (X_1, Y_1 - X_1),$$

$$(X_3, Y_3) = (X_2 + \lambda Y_2, Y_2), \qquad (X_4, Y_4) = (X_3, Y_3 + \beta X_3).$$

It is easy to check that $Y_4 = W$. Therefore we are through.

If $\Psi(X) \in K$, then define

$$(X_1, Y_1) = (X + Y, Y), \qquad (X_2, Y_2) = (X_1, Y_1 - X_1).$$

Then $Y_2 = -X$ and hence $\Psi(Y_2) \in K$. Therefore we are through by applying the previous argument to Y_2.

In view of the previous cases, to complete the proof it is enough to show that if $\deg_Z \Psi(X) = m > 0$ and $\deg_Z \Psi(Y) = n > 0$ then there exists a pair of variables X_1, Y_1 such that

$$(X_1, Y_1) = (X, Y + f(X)) \quad \text{or} \quad (X_1, Y_1) = (X + g(Y), Y)$$

and

$$\deg_Z \Psi(X_1) + \deg_Z \Psi(Y_1) < \deg_Z \Psi(X) + \deg_Z \Psi(Y).$$

We deal with the case $m \le n$. The other case is similar.

By Theorem 2.9, there exists an integer d such that $n = md$. Let μ and ν be the leading coefficients of $\Psi(X)$ and $\Psi(Y)$ respectively. Let $\pi = \nu\mu^{-d}$ and let $(X_1, Y_1) = (X, Y - \pi X^d)$. It is easy to see that the pair (X_1, Y_1) has the required properties.

Thus the proof is complete. □

Lemma 3.6. *Let T be a noetherian ring containing $\mathbb{Q}$. Let $M_1, \ldots, M_l$ be maximal ideals of T and let $J = \bigcap_{i=1}^{l} M_i$. Let L be a finitely generated projective T-module of (constant) rank 1. Let $B = Sym_T(L)$ and $D = B[Y]$ $(= B^{[1]})$. Then $D/JD = (T/J)^{[2]}$. Moreover if $F \in D$ is such that $D/JD = (T/J)[\bar{F}]^{[1]}$, where $\bar{F}$ denotes the image of F in D/JD then there exists a T-algebra automorphism σ of D such that $F \equiv \sigma(Y) \bmod (JD)$.*

Proof. Since T/J is a finite direct of fields, it is easy to see that $D/JD = Sym_{T/J}(L/JL)[Y] = (T/J)^{[2]}$.

Let $T/M_i = K_i$ and let W_i be the image of F in $D/(M_iD)\,(= K_i^{[2]})$ for $1 \le i \le l$. Then $D/JD = (T/J)[\bar{F}]^{[1]}$ implies that $D/(M_iD) = K_i[W_i]^{[1]}$.

Let $N_i = \bigcap_{j \ne i} M_j$. Then $M_i + N_i = T$ for $1 \le i \le l$. In view of Lemmas 3.4 and 3.5, it is easy to see that, for $1 \le i \le l$, there exists a T-algebra automorphism σ_i of D such that

(i) $F \equiv \sigma_i(Y) \bmod (M_iD)$.

(ii) $\sigma_i \equiv \text{identity} \,(\bmod\, N_iD)$.

Let $\sigma = \sigma_l \cdots \sigma_1$. Then $\sigma \equiv \sigma_i \bmod (M_iD)$ for $1 \le i \le l$. Therefore $F \equiv \sigma(Y) \bmod (JD)$. □

Proposition 3.7. *Let T be a noetherian ring containing $\mathbb{Q}$. Let I and J be two proper ideals of T such that $I \subset J \subset \sqrt{I}$. Let L be a finitely generated projective T-module of (constant) rank 1 such that L/IL is free. Let $B = Sym_T(L)$ and $D = B[Y]$ $(= B^{[1]})$. Let $F \in D$ be such that $F \equiv Y \bmod (JD)$. Then there exists a T-algebra automorphism σ of D such that $F \equiv \sigma(Y) \bmod (ID)$.*

Proof. It is easy to see that since $I \subset J \subset \sqrt{I}$ there exists an increasing sequence $I = I_0 \subset I_1 \cdots \subset I_m = J$ of ideals of T and elements $t_1, \ldots, t_m$ of T such that

$I_j = I_{j-1} + (t_j)$ with $t_j^2 \in I_{j-1}$ for $1 \le j \le m$. Therefore it is enough to prove the result in the case $J = I + (t)$ with $t^2 \in I$. Let $\bar{F}, \bar{t}$ denote the images of F and t respectively in D/ID. Then $F \equiv Y \bmod (JD)$ implies that there exists $f \in D/ID$ such that $\bar{F} = Y + \bar{t}f$.

Since L/IL is free (of rank 1) we have $D/ID = (T/I)[X, Y]\ (= (T/I)^{[2]})$ where X is a generator of T/I-module L/IL. Let $f = f_0 + \cdots + f_n$, where f_j is a homogeneous polynomial in X and Y of degree j, $0 \le j \le n$.

Let $\alpha_1, \dots, \alpha_{n+1}$ be distinct rational numbers. Then by Lemma 3.2 there exist elements r_{ij}, $1 \le i \le n+1$, $0 \le j \le n$ of T/I such that

$$f_j = \sum_{i=1}^{n+1} r_{ij}(X + \alpha_i Y)^j, \quad 0 \le j \le n.$$

Let $g_i \in (T/IT)^{[1]}$ be such that

$$g_i(X + \alpha_i Y) = \sum_{j=0}^{n} r_{ij}(X + \alpha_i Y)^j.$$

Then

$$\bar{F} = Y + \bar{t} \sum_{i=1}^{n+1} g_i(X + \alpha_i Y).$$

Let $\alpha_0 = 0$ and $\beta_i = \alpha_i - \alpha_{i-1}$ for $1 \le i \le n+1$. Now for every integer m $(0 \le m \le 2(n+1))$ we define inductively a pair (Z_m, W_m) of variables in $(T/I)[X, Y]\ (= D/ID)$ as follows:

$$(Z_0, W_0) = (X, Y),$$

$$(Z_m, W_m) = \begin{cases} (Z_{m-1} + \beta_i W_{m-1}, W_{m-1}) & \text{if } m = 2i-1, \\ (Z_{m-1}, W_{m-1} + \bar{t} g_i(Z_{m-1})) & \text{if } m = 2i. \end{cases}$$

It is easy to see that since $\bar{t}^2 = 0$ and $\alpha_i = \alpha_{i-1} + \beta_i$, there exist elements $0 = h_1, \dots, h_{n+1} \in D/ID$ such that

$$(Z_m, W_m) = \begin{cases} \left(X + \alpha_i Y + \bar{t}h_i,\ Y + \bar{t} \sum_{k=1}^{i-1} g_k(X + \alpha_k Y)\right) & \text{if } m = 2i-1, \\ \left(X + \alpha_i Y + \bar{t}h_i,\ Y + \bar{t} \sum_{k=1}^{i} g_k(X + \alpha_k Y)\right) & \text{if } m = 2i. \end{cases}$$

In particular $W_{2(n+1)} = Y + \bar{t}\sum_{k=1}^{n+1} g_k(X + \alpha_k Y) = \bar{F}$.

Let $\theta_m : D/ID \to D/ID\ (= (T/I)^{[2]})$ be a T/IT-algebra automorphism defined by $\theta_m(Z_{m-1}) = Z_m$ and $\theta_m(W_{m-1}) = W_m$ for $1 \le m \le 2(n+1)$. Then by Lemma 3.4 (and by induction on m) θ_m can be lifted to a T-algebra automorphism σ_m of D. Let $\sigma = \sigma_{2(n+1)} \cdots \sigma_1$. Then σ is a T-algebra automorphism of D such that $\overline{\sigma(Y)}$ (the image of $\sigma(Y)$ in D/ID) $= W_{2(n+1)} = \bar{F}$.

Thus the proof is complete. □

Theorem 3.8. *Let R be a one-dimensional noetherian domain containing $\mathbb{Q}$ and let A be an $\mathbf{A}^2$-fibration over R. Then there exists $H \in A$ such that A is an $\mathbf{A}^1$-fibration over $R[H]$.*

Proof. By Lemma 3.3, there exists a finite birational extension T of R such that $A \otimes_R T = (Sym_T(L))[Y]$ $(= Sym_T(L)^{[1]})$ for some finitely generated projective T-module L of rank 1. Let $B = Sym_T(L)$ and $D = B[Y] = A \otimes_R T$.

Let I denote the conductor ideal of T over R. If $T = R$ then $A = D = B[Y]$ and in this case taking $H = Y$ we are through. Therefore we assume that $R \neq T$. In that case, since T is finite and birational over R, we have $\mathrm{ht}_R(I) = \mathrm{ht}_T(I) = 1$ and hence $\dim(R/I) = 0$. Therefore $A/IA = (R/I)^{[2]}$.

Let $F \in A$ be such that $A/IA = (R/I)[F']^{[1]}$, where F' denotes the image of F in A/IA. Since $R \hookrightarrow T$ and A is flat over R, we have $A \hookrightarrow D$ $(= A \otimes_R T)$ and therefore we can regard F as an element of D also. Let $J = \sqrt{I}$ in T and let $\bar{F}$ denote the image of F in D/JD. Then it is easy to see that $D/JD = (T/J)[\bar{F}]^{[1]}$.

Since J is an intersection of finitely many maximal ideals of T, by Lemma 3.6 there exists a T-algebra automorphism τ of D $(= (Sym_T(L))[Y])$ such that $F \equiv \tau(Y) \bmod (JD)$. Let $Y_1 = \tau(Y)$ and $L_1 = \tau(L)$. Then, since $D = \tau(D) = (Sym_T(L_1))[Y_1]$, and J is the radical of I in T, by Proposition 3.7 there exists a T-algebra automorphism σ of D such that $F \equiv \sigma(Y_1) \bmod (ID)$.

Since I is the conductor ideal of T over R, $D = A \otimes_R T$ and A is flat over R, it follows that $ID = IA$. Therefore, as $F \in A$ and $F \equiv \sigma(Y_1) \bmod (ID)$, we get $\sigma(Y_1) \in A$. Let $H = \sigma(Y_1)$.

The rest of the proof is devoted to show that A is an $\mathbf{A}^1$-fibration over $R[H]$.

Let P be a prime ideal of R. If $I \not\subset P$ then $R_P = T_P$. Therefore $A_P = D_P = R_P[H]^{[1]}$. If $I \subset P$ then as $F \equiv \sigma(Y_1) \bmod (ID)$ and $H = \sigma(Y_1)$, $IA = ID$ it follows that $F \equiv H \bmod (PA)$. Therefore $A/PA = (R/P)[\bar{H}]^{[1]}$, where $\bar{H}$ denotes the image of H in A/PA. Hence, as $R[H]$ $(= R^{[1]})$ and A are flat over R, A is an $\mathbf{A}^1$-fibration over $R[H]$ by [5, 3.8].

Thus the proof is complete. □

Corollary 3.9. *Let R be a one-dimensional noetherian domain containing $\mathbb{Q}$. Let A be an $\mathbf{A}^2$-fibration over R such that $\Omega_{A/R}$ is extended from R. Then there exists a finitely generated projective R-module L of rank one such that $A = Sym_R(L)^{[1]}$. In particular if $\Omega_{A/R}$ is free then $A = R^{[2]}$.*

Proof. Since A is an $\mathbf{A}^2$-fibration over R, by Theorem 2.4 $\Omega_{A/R}$ is a finitely generated projective A-module of rank 2. Therefore, as $\Omega_{A/R}$ is extended from R and A is faithfully flat over R, there exists a finitely generated projective R-module E of rank 2 such that $A \otimes_R E \cong \Omega_{A/R}$. Since $\operatorname{rank} E = 2 > \dim R$, by a theorem of Serre [3, Corollary 2.7, p. 173] $E \cong L \oplus R$. Hence $\Omega_{A/R} \cong (A \otimes_R L) \oplus A$.

Now by Theorem 3.8, there exists H in A such that A is an $\mathbf{A}^1$-fibration over $R[H]$. Therefore we get the following right exact sequence of A-modules:

$$A \otimes_{R[H]} \Omega_{R[H]/R} \to \Omega_{A/R} \to \Omega_{A/R[H]} \to 0.$$

Since $\Omega_{A/R}$ and $\Omega_{A/R[H]}$ are projective A-modules of rank 2 and 1 respectively and $R[H] = R^{[1]}$, the above sequence is also left exact. Therefore

$$A \otimes_R L \oplus A \cong \Omega_{A/R} \cong \Omega_{A/R[H]} \oplus A$$

showing that $A \otimes_R L \cong \Omega_{A/R[H]}$. Therefore by Proposition 2.7,

$$A = Sym_{R[H]}(R[H] \otimes_R L) = Sym_R(L)^{[1]}.$$

Furthermore if $\Omega_{A/R}$ is free then, it is easy to see that $A \otimes_R L \cong A$ and hence (as R is a retract of A) $L \cong R$. Therefore in this case $A = R^{[2]}$. □

Remark 3.10. If R is a one-dimensional seminormal noetherian domain and A is an $\mathbf{A}^2$-fibration over R, then by Theorems 2.4 and 2.10 it is easy to see that $\Omega_{A/R}$ is extended from R.

Let A be an $\mathbf{A}^r$-fibration over a noetherian ring R. Then by Theorem 2.4 A is a retract of $R^{[n]}$ for some n. The following theorem gives sharp bound on n when $r = 2$ and R is a one-dimensional domain containing $\mathbb{Q}$.

Theorem 3.11. *Let R be a one-dimensional noetherian domain containing $\mathbb{Q}$. Let A be an $\mathbf{A}^2$-fibration over R. Then A is a retract of $R^{[3]}$.*

Proof. By Theorem 3.8, there exists $H \in A$ such that A is an $\mathbf{A}^{[1]}$-fibration over $R[H]$. Therefore $\Omega_{A/R[H]}$ is a finitely generated projective A-module of rank 1. Let

$$\Omega^* = \mathrm{Hom}_A(\Omega_{A/R[H]}, A) \quad \text{and} \quad C = Sym_A(\Omega^*).$$

Then A is a retract of C. Therefore it is enough to prove that $C = R[H]^{[2]} = R^{[3]}$. The rest of the proof is devoted to show that $C = R[H]^{[2]}$.

From the construction it is obvious that C is an $\mathbf{A}^2$-fibration over $R[H]$ such that $\Omega_{C/R[H]}$ is a free C-module of rank 2. Therefore to prove that $C = R[H]^{[2]}$, it is enough to show by Theorem 2.5 that $C \otimes_{R[H]} R_M[H] \cong R_M[H]^{[2]}$ for every maximal ideal M of R. Hence without loss of generality we can assume that R is local.

By Lemma 3.3, there exists a finite birational extension T of R such that $A \otimes_R T = Sym_T(L)^{[1]}$ for some finitely generated projective T-module L of rank 1. Since R is local and T is a finite extension of R, T is semilocal, and hence L is free of rank 1. Therefore $A \otimes_R T = T^{[2]}$. Moreover $A \otimes_R T$ is an $\mathbf{A}^1$-fibration over $T[H]$. Let $D = A \otimes_R T$. Then as in Corollary 3.9 we conclude that

$$\Omega_{D/T} \cong \Omega_{D/T[H]} \oplus (\Omega_{T[H]/T} \oplus_{T[H]} D).$$

Therefore $\Omega_{D/T[H]}$ is a free D-module of rank 1.

Let R' denote the seminormalization of R in T and let $D' = A \otimes_R R'$. Since R' is seminormal in T and R', T are semilocal, it follows from [8, Theorem 9] that the canonical map $Pic(R'^{[m]}) \to Pic(T^{[m]})$ is injective for every positive integer m. Since D' is an $\mathbf{A}^1$-fibration over $R'[H]$, by Theorem 2.4, D' is a retract of $R'[H]^{[n]}$ for some n. Therefore, as $D = D' \otimes_{R'} T$, it follows that the canonical map $Pic(D') \to Pic(D)$ is injective. Hence, as $\Omega_{D/T[H]}$ is free of rank 1, it follows that $\Omega_{D'/R'[H]}$ is free. Hence by Proposition 2.7, $D' = R'[H]^{[1]}$.

Let I denote the conductor ideal of R' over R. Then, since R' is the seminormalization of R in T and R is local it is clear that R' is also local and $(R'/I)_{\text{red}} = (R/I)_{\text{red}} = (R/M) = (R'/N)$, where M and N denote the maximal ideals of the local rings R and R' respectively.

Since $D' = A \otimes_R R' = R'[H]^{[1]}$, it follows that

$$C \otimes_R R' = Sym_A(\Omega^*) \otimes_A D' = Sym_{D'}(\Omega^* \otimes_A D') = R'[H]^{[2]}.$$

Let $C' = C \otimes_R R'$. Let $Y \in C'$ be such that $C' = R'[H][Y]^{[1]}$. Since C is flat over R, we can regard C as a subring of C'. Moreover, as I is the conductor ideal of R' over R with $(R'/I)_{\text{red}} = (R/I)_{\text{red}} = (R/M) = (R'/N)$, from the definitions of C and C' it is clear that (i) $(IC) = (IC')$ and (ii) $(C'/(NC')) = (C/(MC))$.

Let $F \in C$ be such that $F \equiv Y \bmod (NC')$. Then as $C' = R'[H]^{[2]} = R'[H]^{[1]}[Y]$ by Proposition 3.7 there exists an $R'[H]$-algebra automorphism σ of C' such that $F \equiv \sigma(Y) \bmod (IC')$. Since $F \in C$ and $(IC) = (IC')$, we get that $\sigma(Y) \in C$. Let $G = \sigma(Y)$.

Claim. *C is an $\mathbf{A}^1$-fibration over $R[H, G]$.*

Assume the Claim for the time being. Since

$$\Omega_{C/R[H]} = (C \otimes_{R[H,G]} \Omega_{R[H,G]/R[H]}) \oplus \Omega_{C/R[H,G]}$$

and $\Omega_{C/R[H]}$ is a free C-module of rank 2 it follows that $\Omega_{C/R[H,G]}$ is free of rank 1. Therefore in view of the Claim and Proposition 2.7, we get that $C = R[H, G]^{[1]} = R[H]^{[2]}$.

Thus the proof of the theorem will be complete if we prove the Claim.

Proof of the Claim. Since $C' = C \otimes_R R' = R'[H, G]^{[1]}$ and $R/M = R'/N$, we get that $C/(MC) = C'/(NC') = R/M[\bar{H}, \bar{G}]^{[1]}$, where $\bar{H}$, $\bar{G}$ denote the images of H, G respectively in $C/(MC)$. Therefore, as R and R' are birational and C, $R[H, G]$ are flat over $R[H]$ by [5, 3.8], C is an $\mathbf{A}^1$-fibration over $R[H, G]$.

Thus the Claim and hence Theorem 3.11 is proved. □

We conclude this paper by giving the following example of an *indecomposable* $\mathbf{A}^2$-fibration over a one-dimensional noetherian domain R containing $\mathbb{Q}$. In view of Corollary 3.9 and Remark 3.10 it is clear that such a ring R ought not to be seminormal. This example shows that Theorem 3.8 is (in some sense) best possible for an arbitrary domain R.

Example 3.12. Let K be a field of characteristic 0. Let

$$T = K[Z]\ (= K^{[1]}), \qquad R = K[Z^2, Z^3],$$

$$D = T[X, Y]\ (= T^{[2]}), \qquad A = R[X, Y + Z(X^2Y^2)] + (Z^2D).$$

Claim. *A is an indecomposable $\mathbf{A}^2$-fibration over R.*

Proof. We first prove that A is an $\mathbf{A}^2$-fibration over R.

Let $I = (Z^2T) = ((Z^2, Z^3)R)$. Then I is the conductor ideal of T over R. Therefore the following square of rings is cartesian:

$$\begin{array}{ccc} R & \hookrightarrow & T \\ \downarrow & & \downarrow \\ R/I & \hookrightarrow & T/I \end{array}$$

Moreover A is the pullback of T-algebra $D = T[X, Y]$ and R/I-algebra $(R/I)[W_1, W_2]$ through T/I-algebra isomorphism σ defined by

$$\sigma(W_1) = X, \qquad \sigma(W_2) = Y + (\bar{Z}X^2)Y^2,$$

where $\bar{Z}$ denotes the image of Z in T/I. Therefore by [10, 3.1], A is a retract of a polynomial algebra over R. Hence A is finitely generated and flat over R. Now, as $A \otimes_R T = D$ and $A/(IA) \cong (R/I)[W_1, W_2]$ it is easy to see that A is an $\mathbf{A}^2$-fibration over R.

Now we show that A is indecomposable.

If A is decomposable then there exist $\mathbf{A}^1$-fibrations B and C over R such that $A = B \otimes_R C$ as R-algebras.

Since B is an $\mathbf{A}^1$-fibration over R, it is clear that B is the pullback of T-algebra $T[U]$ and R/I-algebra $R/I[W]$ through T/I-algebra isomorphism θ defined by $\theta(W) = U + \bar{Z}h(U)$, where $\bar{Z}$ denotes the image of Z in T/I and $h(U) \in (T/I)[U]$. Similar considerations hold for C.

From the above discussion it is obvious now that if $A = B \otimes_R C$ then there exist U, $V \in D$ and $f(U) \in T[U]$, $g(V) \in T[V]$ such that

$$D = T[X, Y] = T[U, V] \quad \text{and} \quad A = R[U + Zf(U), V + Zg(V)] + (Z^2D).$$

Let $\overline{f(U)}, \overline{g(V)}$ denote the images of $f(U)$, $g(V)$ in $D/(ID)$. Since $D/(ID) = (K[Z]/(Z^2))[U, V]$, it follows that there exist $f_1, g_1 \in K^{[1]}$ such that $\overline{Zf(U)} = \bar{Z}f_1(U)$ and $\overline{Zg(V)} = \bar{Z}g_1(V)$. Let

$$U_1 = U + \bar{Z}f_1(U), \qquad V_1 = V + \bar{Z}g_1(V),$$

$$X_1 = X, \qquad Y_1 = Y + \bar{Z}X^2Y^2.$$

Then $A/(IA) = K[X_1, Y_1] = K[U_1, V_1] = K^{[2]}$ and $D/(ID) = (T/I)[X, Y] = (T/I)[U, V] = (T/I)[U_1, V_1] = (T/I)[X_1, Y_1]$. Therefore in $D/(ID)$ we have

$$1 + 2\bar{Z}X^2Y = \mathbf{J}_{(X,Y)}(X_1, Y_1) = \mathbf{J}_{(U_1,V_1)}(X_1, Y_1)\mathbf{J}_{(U,V)}(U_1, V_1)\mathbf{J}_{(X,Y)}(U, V).$$

Since $A/(IA) = K[X_1, Y_1] = K[U_1, V_1]$ and $D = T[X, Y] = T[U, V]$, it follows that $\mathbf{J}_{(U_1,V_1)}(X_1, Y_1)$ and $\mathbf{J}_{(X,Y)}(U, V) \in K^*$. Moreover

$$\mathbf{J}_{(U,V)}(U_1, V_1) = 1 + \bar{Z}(f_1'(U) + g_1'(V)).$$

Therefore in $D/(ZD) = K[X, Y] = K[U, V]$ we have $2(X^2Y) = f_1'(U) + g_1'(V)$. This is a contradiction by Lemma 3.1.

Thus the proof is complete. □

Remark 3.13. Note that by suitable reductions (and appropriate modifications in the proofs) it is easy to see that Theorems 3.8 and 3.11 and Corollary 3.9 will be true for any arbitrary one-dimensional noetherian ring R containing $\mathbb{Q}$ which is not necessarily a domain.

Remark 3.14. The proof of Theorem 3.11 essentially shows that: *If R is a one-dimensional noetherian ring containing $\mathbb{Q}$, $S = R^{[n]}$ and A is an $\mathbb{A}^1$-fibration over S, then A is a retract of $S^{[2]}$.*

References

[1] S.S. Abhyankar and T.T. Moh, Embeddings of the line in the plane, J. Reine Angew. Math. 276 (1975) 148–166.

[2] T. Asanuma, Polynomial fibre rings of algebras over noetherian rings, Invent. Math. 87 (1987) 101–127.

[3] H. Bass, Algebraic K-theory (Benjamin, New York, 1968).

[4] H. Bass, E.H. Connell and D.L. Wright, Locally polynomial algebras are symmetric algebras, Invent. Math. 38 (1977) 279–299.

[5] S.M. Bhatwadekar and A.K. Dutta, On $\mathbb{A}^1$-fibrations of subalgebras of polynomial algebras, Compositio Math., to appear.

[6] S.M. Bhatwadekar and A. Roy, Some theorems about projective modules over polynomial rings, J. Algebra 86 (1984) 150–158.

[7] E. Hamann, On the R-invariance of $R[X]$, J. Algebra 35 (1975) 1–16.

[8] F. Ischebek, On the Picard group of polynomial rings, J. Algebra 88 (1984) 395–404.

[9] A. Sathaye, Polynomial ring in two variables over a D.V.R.: a criterion, Invent. Math. 74 (1983) 159–168.

[10] J. Yanik, Projective algebras, J. Pure Appl. Algebra 21 (1981) 339–358.